An optimization method for approximating the macroscopic strength criterion of stone column reinforced soils

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AN OPTIMIZATION METHOD FOR APPROXIMATING THE MACROSCOPIC STRENGTH CRITERION OF STONE COLUMN REINFORCED SOILS

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ABSTRACT: *In this contribution, the yield design homogenization method is applied to the evaluation of the ultimate bearing capacity of a purely cohesive soil reinforced by a periodic array of columnar inclusions, made of a purely frictional material (stone column technique). The method is implemented following a three-step procedure. a) First, the numerical determination of the macroscopic strength criterion is performed using the kinematic approach of yield design. b) Second, an easier to handle formulation of the criterion is obtained as the sum of a few ellipsoids in the stress space. c) Finally, the so-obtained approximation is incorporated into a numerical code, leading to the determination of an optimized upper bound for the ultimate load bearing capacity of a reinforced soil foundation, which is compared with previously obtained estimates for the same problem.*

1 INTRODUCTION

The growing need of constructions on poor quality soils has fostered the development of various soil improvement methods, among which the so-called ”stone column” reinforcement technique concerned by this contribution. This technique generally consists in incorporating into a purely cohesive native soil, a periodic distribution of cylindrical inclusions made of a highly frictional material (Priebe 1995). One of the main objectives of such a reinforcement technique is to increase the ultimate bearing capacity of a foundation soil subject to vertical loading. A key ingredient to such a calculation is the prior determination of the macroscopic strength criterion of the reinforced soil regarded as a homogenized material.

Using a homemade limit analysis code, this contribution presents a numerical upper bound estimate for the criterion in the stress space, for a given reinforcement volume fraction, soil cohesion and reinforcement friction angle (section 2). In view of being able to analyze the stability of a stone column-reinforced structure, a numerical optimization procedure is proposed based on the use of convex ellipsoidal sets defined by few parameters, which are combined in such a way as to obtain a fairly accurate approximation to the criterion (section 3). In section 4, the whole procedure is illustrated on producing an upper bound kinematic estimate for the ultimate bearing capacity of a stone column reinforced foundation.

2 AN IMPROVED UPPER BOUND FOR THE MACROSCOPIC STRENGTH CRITERION OF REINFORCED SOIL

2.1 Problem statement

A stone column reinforced soil may be seen as a periodic composite material, made of a regular array of cylindrical columnar inclusions embedded into the soil mass (see Figure 1). The reinforcement layout may be entirely described by a unit cell $\mathcal{C}$ of side s (spacing between two neighboring columns) containing one single reinforcing column of radius ρ. The reinforcement volume fraction η, which is the ratio between the volume occupied by the column and the volume of the unit cell, is then defined as:

$$\eta = \frac{\pi \rho^2}{s^2} \tag{1}$$

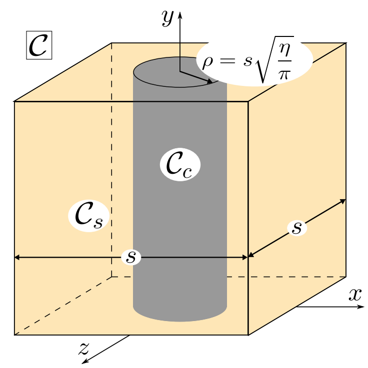


Fig. 1. Representative unit cell for a stone column reinforced soil.

The native soil is taken here as a purely cohesive soft clay, its strength capacities being described by a von Mises yield condition of the form:

$$f^s(\underline{\underline{\sigma}}) = \sqrt{\frac{1}{2}\underline{\underline{s}} : \underline{\underline{s}}} - k \leq 0 \tag{2}$$

where $\underline{\underline{s}}$ is the deviatoric stress and k the yield strength under pure shear conditions.

The column constituent material is a purely frictional granular soil obeying a Drucker-Prager strength criterion of the form:

$$f^c(\underline{\underline{\sigma}}) = \sqrt{\frac{1}{2}\underline{\underline{s}} : \underline{\underline{s}}} + a(\varphi)\mathrm{tr}\underline{\underline{\sigma}} \leq 0 \tag{3}$$

where φ denotes the friction angle. The expression of a is chosen as a function of φ in such a way that, under plane strain conditions, the Drucker-Prager criterion coincides with the classical Mohr-Coulomb criterion associated with the same friction angle. Calculations will be carried out with a perfect bonding condition at the soil-column interface.

2.2 Macroscopic strength criterion

Referring to this yield design (or limit analysis) theory, the macroscopic strength domain $\mathbf{G}^{hom}$ and the related strength criterion F of the reinforced soil are defined as follows:

$$\underline{\underline{\Sigma}} \in \mathbf{G}^{hom} \;\Leftrightarrow\; F\left(\underline{\underline{\Sigma}}\right) \leq 0 \;\Leftrightarrow\; \begin{cases} \forall \underline{\underline{\sigma}} \text{ statically admissible with } \underline{\underline{\Sigma}} \\ \forall \underline{\xi} \in \mathcal{C}_\alpha,\; f^\alpha\left(\underline{\underline{\sigma}}^\alpha\left(\underline{\xi}\right)\right) \leq 0, \quad \alpha = s, c \end{cases} \tag{4}$$

where $\mathcal{C}_\alpha$ is the unit cell sub-domain occupied by constituent α (c for column or s for soil) and $f^\alpha(.)$ its yield strength function given by either (2) or (3). A stress field $\underline{\underline{\sigma}}$ is statically admissible with a macroscopic stress $\underline{\underline{\Sigma}}$ if it complies with the following conditions:

- $\underline{\underline{\sigma}}$ is in equilibrium with no body forces:

$$\mathrm{div}\underline{\underline{\sigma}} = 0 \tag{5}$$

- the stress vector remains continuous across any possible discontinuity surfaces of the stress field:

$$\left[\underline{\underline{\sigma}}\right] \cdot \underline{n} = 0 \tag{6}$$

  where $\left[\underline{\underline{\sigma}}\right]$ denotes the jump of $\underline{\underline{\sigma}}$ across such a surface following its unit normal $\underline{n}$.
- $\underline{\underline{\sigma}} \cdot \underline{n}$ is *anti-periodic*, which means that it takes opposite values at any couples of points located on the opposite sides of the unit cell.
- $\underline{\underline{\Sigma}}$ is equal to the volume average of $\underline{\underline{\sigma}}$ over the unit cell:

$$\underline{\underline{\Sigma}} = \frac{1}{|\mathcal{C}|} \int_{\mathcal{C}} \underline{\underline{\sigma}} \, \mathrm{d}\mathcal{C} = \langle \underline{\underline{\sigma}} \rangle \tag{7}$$

2.3 *Kinematic definition*

According to the periodic homogenization method, a velocity field kinematically admissible with a macroscopic strain rate tensor $\underline{\underline{D}}$ is given, up to a rigid body motion, by:

$$\forall \underline{\xi} \in \mathcal{C}, \; \underline{U}\left(\underline{\xi}\right) = \underline{\underline{D}} \cdot \underline{\xi} + \underline{\nu}\left(\underline{\xi}\right) \tag{8}$$

where $\underline{\nu}\left(\underline{\xi}\right)$ is a *periodic* fluctuation. The dualisation of the equilibrium equations by means of the *virtual work principle* with strength conditions (2) and (3), leads to the following kinematic definition of the macroscopic strength domain (Suquet 1985; de Buhan 1986):

$$\mathbf{G}^{hom} = \bigcap_{\underline{\underline{D}} \in \mathbb{R}^6} \left\{ \underline{\underline{\Sigma}} \,|\, \underline{\underline{\Sigma}} : \underline{\underline{D}} \leq \pi^{hom}\left(\underline{\underline{D}}\right) \right\} \tag{9}$$

with

$$\pi^{hom}\left(\underline{\underline{D}}\right) = \min_{\underline{U} \text{ ka } \underline{\underline{D}}} \left\{ \langle \pi\left(\underline{\underline{d}}\right) \rangle = \frac{1}{|\mathcal{C}|} \sum_\alpha \int_{\mathcal{C}_\alpha} \pi^\alpha\left(\underline{\underline{d}}\right) \, \mathrm{d}\mathcal{C}_\alpha \right\} \tag{10}$$

where $\underline{\underline{d}}$ denotes the microscopic strain rate tensor field associated to the velocity field $\underline{U}$, which is assumed to remain continuous, and $\pi^\alpha\left(\underline{\underline{d}}\right)$ is the support function of constituent α, given by:

$$\pi^s\left(\underline{\underline{d}}\right) = \begin{cases} k\sqrt{2\underline{\underline{d}} : \underline{\underline{d}}} & \text{if } \mathrm{tr}\underline{\underline{d}} = 0 \\ +\infty & \text{otherwise} \end{cases} \tag{11}$$

for the purely cohesive soil, and

$$\pi^c\left(\underline{\underline{d}}\right) = \begin{cases} 0 & \text{if } \mathrm{tr}\underline{\underline{d}} \geq 3\sqrt{2}a\sqrt{\underline{\underline{d}} : \underline{\underline{d}} - \frac{1}{3}\left(\mathrm{tr}\underline{\underline{d}}\right)^2} \\ +\infty & \text{otherwise} \end{cases} \tag{12}$$

for the purely frictional column material.

2.4 Numerical simulation and results

An upper bound estimate to the previously defined macroscopic strength domain is carried out by a homemade limit analysis code (similar to that developed by Makrodimopoulos & Martin (2007) or Pastor et al. (2011)) and using the second-order cone programming solver MOSEK (2008). Calculations are performed on the reinforced soil unit cell $\mathcal{C}$ leading to the evaluation of limit loads under plane strain conditions in the (x, y)-plane. This means that the unit cell is subject to a macroscopic strain rate tensor of the form:

$$\underline{\underline{D}} = \underline{\underline{\Delta}}(\gamma, \delta) = \begin{pmatrix} \cos\gamma\cos\delta & \frac{1}{2}\sin\delta & 0 \\ \frac{1}{2}\sin\delta & \sin\gamma\cos\delta & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

where angles γ and δ specify the orientation of the loading in the space of plane strains in the Oxy-plane (see Figure 2(a)).

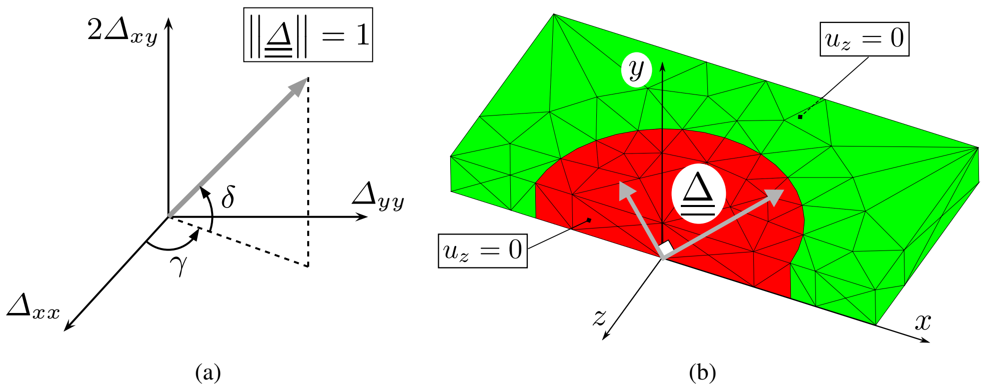


Fig. 2. Plane strain loading of the unit cell: (a) angular parametrization, (b) mesh adopted for the numerical optimization.

As described by Hassen et al. (2013), thanks to the geometrical, material and loading symmetries, calculations may be performed on one half of the unit cell with a smooth contact condition on the sides parallel to the Oxy-plane. It can be also proved that the analysis can be restricted to a ”slice” in the direction y. The mesh chosen to modelize the problem is given in Figure 2(b) with ten noded tetrahedric elements. Here the maximal resisting work of the unit cell will be minimized using, in each element, a velocity field chosen as a quadratic function of the coordinates, so that the strain field will be linear.

The loading is prescribed by applying *periodicity* conditions to the sides normal to the Ox-axis as well as to the sides normal to the Oy-axis according to equation (8). For each couple of angles (γ, δ), the numerical evaluation of the support function (denoted by $\pi^{num}(\underline{\underline{\Delta}})$) is stored verifying:

$$\pi^{hom}(\underline{\underline{\Delta}}) \leq \pi^{num}(\underline{\underline{\Delta}}), \forall \underline{\underline{\Delta}} \tag{14}$$

As an illustrative example, the so-obtained macroscopic strength criterion has been drawn in the space of non-dimensional macroscopic stresses $(\frac{\Sigma_{xx}}{k}, \frac{\Sigma_{yy}}{k}, \frac{\Sigma_{xy}}{k})$ for the following typical values:

$$\eta = 28\%, \quad \varphi = 35° \tag{15}$$

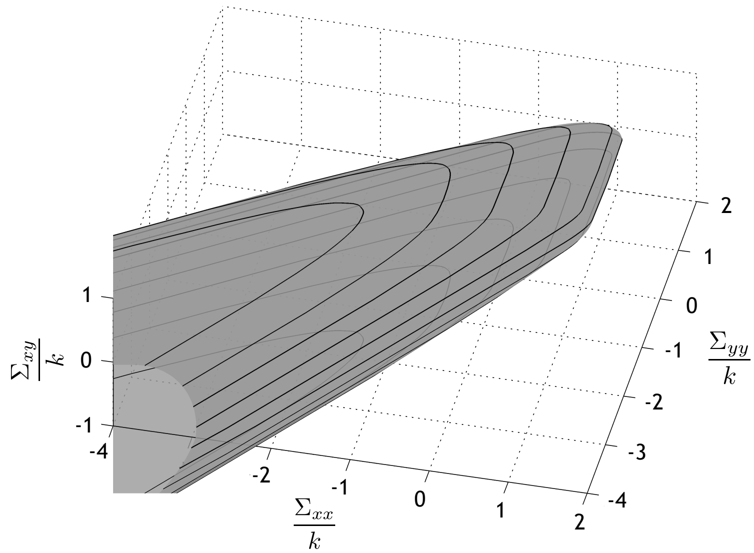


Fig. 3. Macroscopic strength domain in the $(\frac{\Sigma_{xx}}{k}, \frac{\Sigma_{yy}}{k}, \frac{\Sigma_{xy}}{k})$-space.

Figure 3 depicts the macroscopic yield surface and figure 4 displays its cross section by the zero shear stress plane, as well as those corresponding to the soil and the column materials. The latter figure clearly shows an improvement of the soil strength for high compressive stress states, due to the column reinforcement.

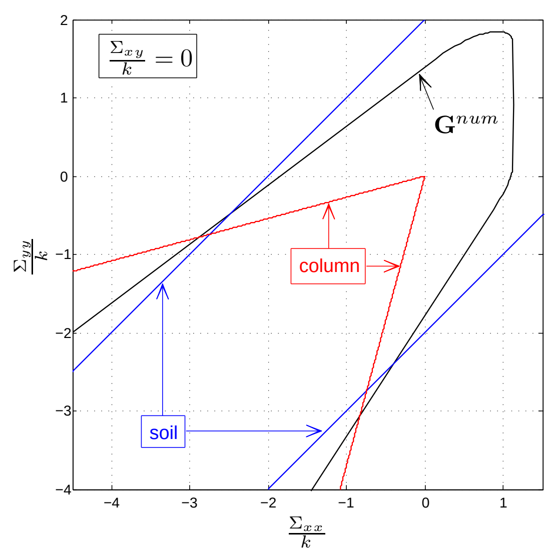


Fig. 4. Representations in the $(\frac{\Sigma_{xx}}{k}, \frac{\Sigma_{yy}}{k})$-plane of the native soil, reinforcement and reinforced soil strength domains.

It is worth noting that for many couples (γ, δ), there is no limit loads, so that the support value becomes infinite. The set of macroscopic strains directions generating a limit load is a cone denoted by $\{\underline{\underline{\Delta}}\}^{num}$. Moreover, despite the fact that the column is cohesionless, the reinforced material has a significant strength in the region of tensile stresses, the soil embedding the column.

3 APPROXIMATION USING CONVEX ELLIPSOIDAL SETS

3.1 Theoretical principle

As it will be shown in the next section, using the so-obtained strength criterion for computing limit loads of reinforced soil structures proves to be highly complex. In order to approximate the criterion with the fewest parameters as possible, it has been decided to use sums of convex ellipsoidal sets, in the sense of Minkowski summation. This means for instance that, for two sets A and B, the Minkowski sum (symbolised by $\oplus$) is defined as:

$$A \oplus B = \{a + b \,|\, a \in A, b \in B\} \tag{16}$$

which is sketched in figure 5, where A and B are a triangle and a circle, respectively.

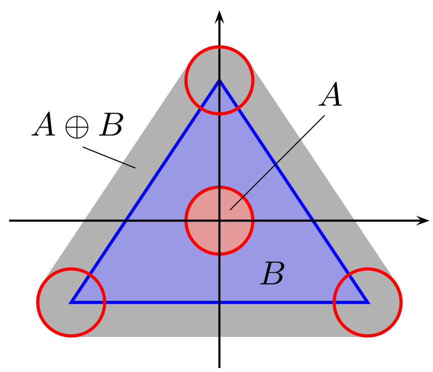


Fig. 5. Minkowski sum (shaded area) of two sets (triangle and circle).

Since, the macroscopic strength criterion is numerically defined as a polytope in the space of non-dimensional macroscopic stresses $(\frac{\Sigma_{xx}}{k}, \frac{\Sigma_{yy}}{k}, \frac{\Sigma_{xy}}{k})$, it is well-founded to try to approximate this polytope by a Minkowski sum of convex sets. It has been decided to use ellipsoidal sets, the support function of which is defined as (Bleyer & de Buhan 2013):

$$\pi_{ell}(\underline{\underline{\Delta}}) = \sup_{f_{ell}(\underline{\underline{\Sigma}})\leq 0} \underline{\underline{\Sigma}} : \underline{\underline{\Delta}} = \sqrt{{}^t\underline{\underline{\Delta}} : \mathbb{A} : \underline{\underline{\Delta}}} - {}^t\underline{\underline{c}} : \underline{\underline{\Delta}} \tag{17}$$

where $\mathbb{A}$ is a fourth order tensor which specifies the orientation and dimensions of the ellipsoid in the stress space, $\underline{\underline{c}}$ the coordinates of its center and $f_{ell}(.)$ the yield function of the ellipsoid. For a sum of N ellipsoidal sets, the support function writes:

$$\pi_{Nell}(\underline{\underline{\Delta}}) = \sum_k \sqrt{{}^t\underline{\underline{\Delta}} : \mathbb{A}_k : \underline{\underline{\Delta}}} - {}^t\underline{\underline{c}} : \underline{\underline{\Delta}} \tag{18}$$

where $\mathbb{A}_k$ is the fourth order tensor defining ellipsoid number k and $\underline{\underline{c}}$ defines the center of one of them.

To approximate the strength domain, the standard deviation of the difference between π^{num} and π_{Nell} is minimized along all the macroscopic strains located in the cone $\{\underline{\underline{\Delta}}\}^{num}$. In order to obtain an upper bound for $\mathbf{G}^{hom}$, the support function of the approximation must remain larger than the numerical one for all strain directions of $\left\{\underline{\underline{\Delta}}\right\}^{num}$. The minimization problem may therefore be written as follows:

$$\min_{\underline{x}\in\mathbb{R}^{6N+3}} \sum_{\underline{\underline{\Delta}}\in\left\{\underline{\underline{\Delta}}\right\}^{num}} \left(\pi^{num}(\underline{\underline{\Delta}}) - \pi_{Nell}(\underline{x},\underline{\underline{\Delta}})\right)^2 \tag{19}$$

under the condition

$$\pi^{num}(\underline{\underline{\Delta}}) \leq \pi_{Nell}(\underline{x},\underline{\underline{\Delta}}), \quad \forall\underline{\underline{\Delta}} \in \left\{\underline{\underline{\Delta}}\right\}^{num} \tag{20}$$

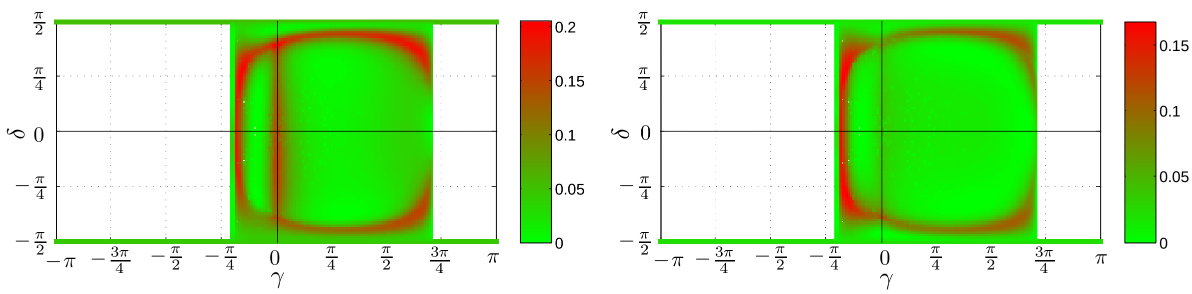
where vector $\underline{x}$ contains the $6N$ components associated to the tensors $\mathbb{A}_k$ $(k = 1,\dots,N)$ as well as the three components of $\underline{\underline{c}}$.

3.2 Application to the macroscopic strength criterion

The approximation is made here using a sum of 3 and 8 ellipsoids. It is possible to evaluate the relative error due to the approximation for all strain directions of $\left\{\underline{\underline{\Delta}}\right\}^{num}$. For the stone column reinforced soil, the cone of outer normals to the strength domain is:

$$\underline{\underline{\Delta}} \in \left\{\underline{\underline{\Delta}}\right\}^{num} \Leftrightarrow \underline{\underline{\Delta}} = \sum_i \beta_i \underline{\underline{\Delta}}_i^{lim}, \forall\beta_i \geq 0 \tag{21}$$

where $\underline{\underline{\Delta}}_i^{lim}$ are the extremal strain directions normal to the strength domain, which can be detected numerically.



(a) (b)

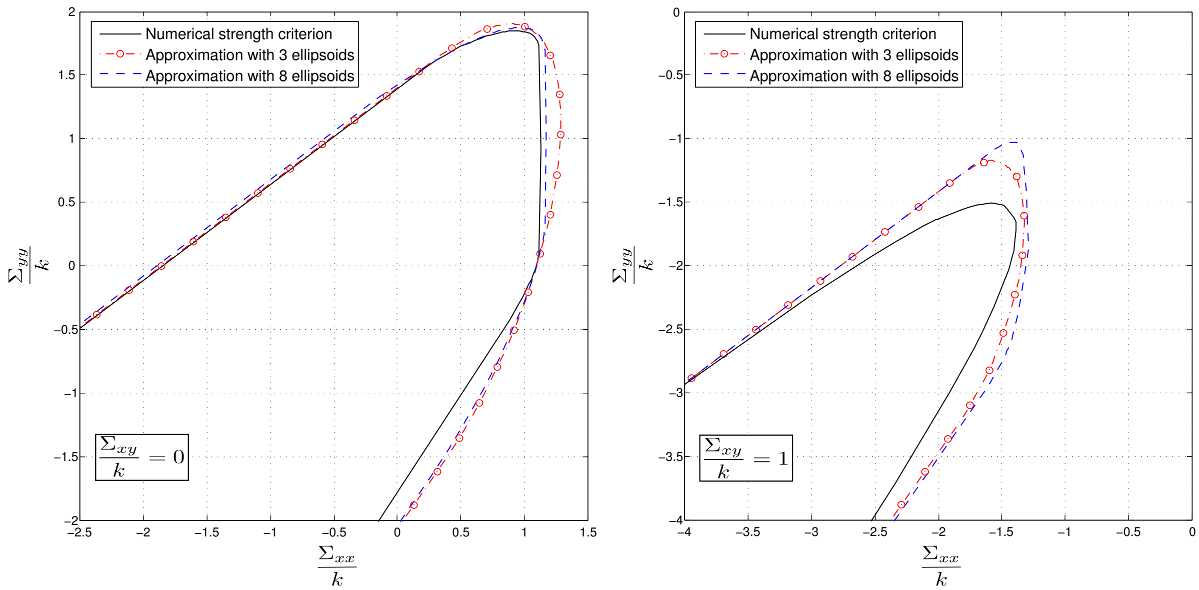
Fig. 6. Approximation error (colorbar) depending on the strain orientation (γ,δ): (a) sum of 3 ellipsoids, (b) sum of 8 ellipsoids.

Figure 6 displays this error for both approximations as function of the strain orientation (γ,δ). In the case considered here, the maximum value of this relative error reaches here 20.4% for the approximation by a sum of 3 ellipsoids, decreasing to 16.8% for 8 ellipsoids.

The so-obtained approximations are also represented in the $(\frac{\Sigma_{xx}}{k},\frac{\Sigma_{yy}}{k})$-plane in Figure 7 for the two extreme values of $\frac{\Sigma_{xy}}{k} = 0$ and 1. As expected both approximations are upper bounds for the macroscopic strength criterion. The average value of the relative error is reduced from 4.96% for 3 ellipsoids to 2.86% for 8 ellipsoids.

4 APPLICATION TO THE DETERMINATION OF THE LOAD BEARING CAPACITY OF A STONE COLUMN REINFORCED FOUNDATION

In order to apply the previously described procedure to a design example, the case of a strip footing acting upon a semi-infinite purely cohesive soil reinforced by purely frictional stone



(a) (b)

Fig. 7. Representations of strength criterion and approximations: (a) $\frac{\Sigma_{xy}}{k} = 0$, (b) $\frac{\Sigma_{xy}}{k} = 1$.

columns is considered (see Figure 8). The characteristics for the reinforcement are given by (15) and the specific weight will first not be taken into account, then set equal to the same value $\gamma = \gamma^s = \gamma^c = 18$ kN/m^3. The ultimate bearing capacity denoted by Q^+ will be estimated by considering that the whole soil layer has been reinforced.

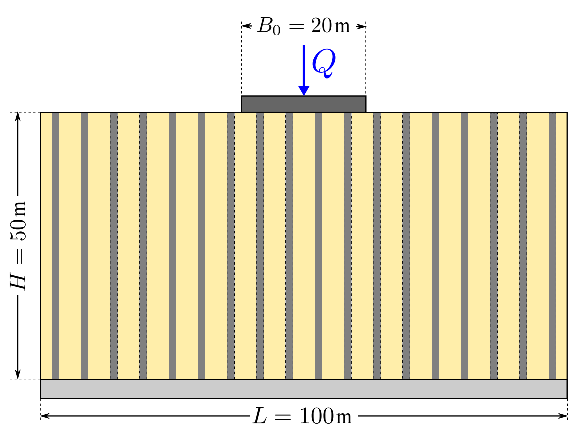


Fig. 8. Ultimate bearing capacity analysis of a stone column reinforced foundation.

4.1 A simple failure mechanism

The first upper bound estimate is searched by using a failure mechanism made of two rectangular triangular blocks, involving three velocity discontinuity lines. The two blocks are characterized by an angle α_i with a velocity $\underline{U}_i$ inducing a velocity inclined at β_i where i is the number of the block. The velocity jump between the two blocks is noted $[\underline{U}]_1^2$ and is inclined at an angle β_1^2 as sketched in Figure 9.

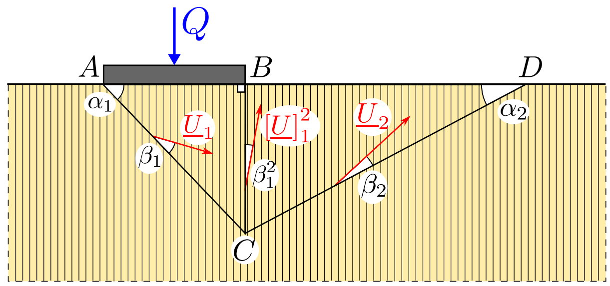


Fig. 9. Rigid block failure mechanism used for a first upper bound estimate.

It can be proved that the virtual work of external forces in such a failure mechanism may be written as:

$$W_e = QU_1 \sin(\alpha_1 - \beta_1) + \gamma \left(U_1 \frac{\tan \alpha_1}{2} - U_2 \frac{\tan^2 \alpha_1}{2 \tan \alpha_2} \frac{\sin(\alpha_2 + \beta_2)}{\sin(\alpha_1 - \beta_1)} \right) \tag{22}$$

where U_i denote the norm of the velocities. Since only velocity discontinuities are involved in the mechanism, the maximum resisting work is expressed as:

$$W_{mr} = \int_{AC} \pi^{num}(\underline{n}_{AC}, \underline{U}_1)\,\mathrm{d}s + \int_{BC} \pi^{num}(\underline{n}_{BC}, [\underline{U}]_1^2)\,\mathrm{d}s + \int_{CD} \pi^{num}(\underline{n}_{CD}, \underline{U}_2)\,\mathrm{d}s \tag{23}$$

with $\underline{n}$ is the normal unit vector to each discontinuity line and $\pi^{num}(\underline{n}, \underline{U})$ is the support function of the velocity jump accross this line. The value of $\pi^{num}(\underline{n}, \underline{U})$ is obtained from $\pi^{num}(\underline{\underline{\Delta}})$ by a reasoning explained in details by Hassen et al. (2013).

The application of the kinematic approach of yield design leads to the following inequations:

$$Q \leq Q^+ \ \Rightarrow \ \forall \underline{U},\, W_e(\underline{U}) \leq W_{rm}(\underline{U}) \tag{24}$$

which, using (22) and (23), allows to express the upper bound estimate $Q^{ub}_{2blocks}$ for this failure mechanism. The minimization is made by varying the angular parameters α_i and β_i numerically. The results are reported in Table 1 for the weightless problem and for $\gamma = 18$ kN/m^3.

4.2 Numerical application of the approximation of the macroscopic strength criterion

The approximations of the macroscopic strength criterion, introduced in section 3.2, will be now used for treating the same problem. This problem can be solved considering one half of the structure, with smooth contact boundary conditions for the lateral sides of the model as well as for the bottom side. The optimization of the problem is carried out by a homemade limit analysis code using the conic problem optimizer MOSEK. The mesh used for these calculations is displayed in Figure 10 with six noded elements.

For both approximations of the macroscopic strength criterion, the upper bound estimate (noted Q^{ub}_{Nell}) and the associated failure mechanism are obtained. All results are given in Table 1 and Figure 11 represents the failure mechanism with an approximation using 8 ellipsoids taken into account the gravity or not. The influence of gravity is reflected in the fact that the extension of the mechanism is larger when the specific weight is not taken into account.

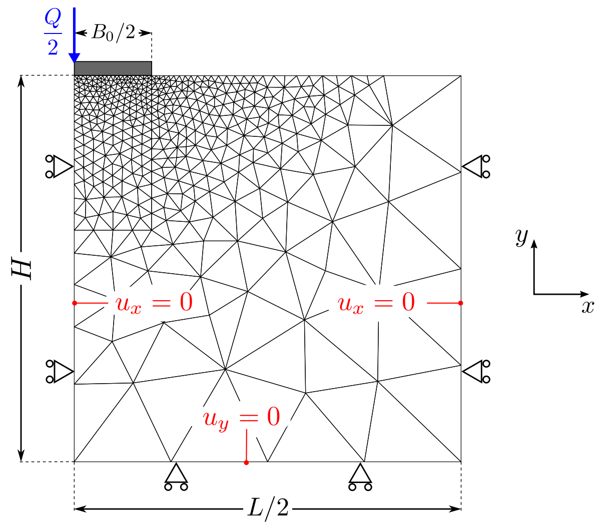
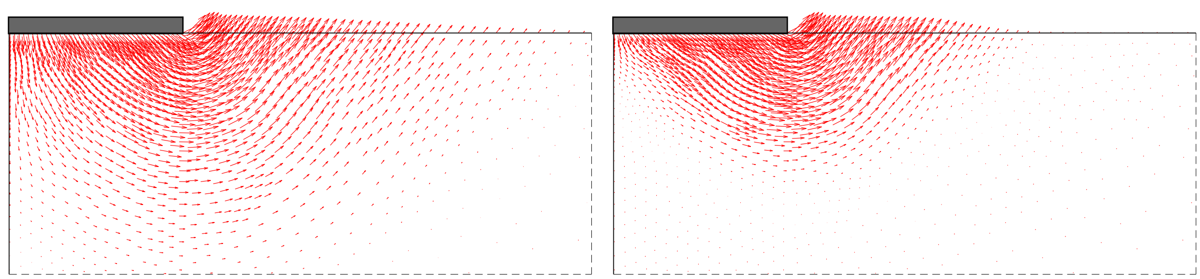


Fig. 10. Adopted finite element mesh for upper bound estimate with ellipsoidal set approximation.



(a) (b)

Fig. 11. Failure mechanism using the 8 ellipsoids approximation: (a) Without gravity, (b) Gravity loaded.

Finally, different upper bound estimates previously obtained are summed up in Table 1. These values are compared to a lower bound estimate of Q^+ obtained from elastoplastic calculations by Hassen et al. (2010) and denoted here by $Q^{lb}_{el.pl.}$. The different percentages reported in this table represent the relative difference between each of these bounds and the well-known exact value for a non-reinforced purely cohesive soil $Q^+_{nr}/kB_0 = \pi + 2 \simeq 5.14$.

Table 1. Upper and lower bounds estimates for the bearing capacity of stone columns reinforced soils.

Case	without gravity (gap %)	gravity loaded (gap %)
$\frac{Q^{lb}_{el.pl}}{kB_0}$	5.95 (15.72)	6.30 (22.53)
$\frac{Q^{ub}_{2blocks}}{kB_0}$	7.36 (43.15)	7.79 (51.51)
$\frac{Q^{ub}_{3ell}}{kB_0}$	6.66 (29.53)	7.30 (41.98)
$\frac{Q^{ub}_{8ell}}{kB_0}$	6.63 (28.95)	7.27 (41.40)

As expected, the upper bounds obtained by ellipsoid-based approximations are closer to the lower bounds than the result of the kinematic approach using blocks failure mechanisms. The potential loss of accuracy induced by the approximation is completely offset by the possibility of using more complicated failure mechanisms. Thanks to the present method, the difference between bounds of the non-dimensionnal load value $\frac{Q^+}{kB_0}$, is equal to 0.68 without gravity (1.41 with two blocks failure mechanism) and 0.97 when gravity is accounted for (1.39 with two blocks failure mechanism).

5 CONCLUDING REMARKS

The present contribution makes use a numerical code to evaluate the macroscopic strength criterion of a stone-column reinforced soil. Owing to the kinematic approach of yield design, this evaluation is a rigorous upper bound, depending on the material and geometrical characteristics. An analytical approximation of the macroscopic criterion as a sum of ellipsoids is then developed, aimed at performing calculations on reinforced geotechnical structures. The performance of such an approximation is then proved on a classical geotechnical design problem. This represents a fully original approach for treating geotechnical engineering problems, with acceptable computation times and could be developed as an alternative to finite element calculation with elastoplastic procedure.

The reinforcement by columnar inclusions provides, as expected, a gain in terms of bearing capacity, comparatively to the native soil. This improvement magnified when gravity is taken into account, due to the strength provided by the frictional material for highly compressive stresses. The efficiency of such a reinforcement technique is confirmed and could be measured for other typical geotechnical problems, as stability analysis of an embankment resting upon a reinforced soil or a buried foundation.

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